Images of induced endomorphisms in Ext (H, G)

By S. G. TELLMAN in Claremont (California, U.S.A.)

We give a homological proof for [1, Theorem 62.4] and point out a dual result. All groups considered are abelian; we use the notation of [1].

Theorem A [1, Theorem 62. 4]. *Let x denote the element of* Ext(H, G) *determined by the exact sequence*

$$0 \longrightarrow G \xrightarrow{\alpha} E \xrightarrow{\beta} H \longrightarrow 0;$$

let $\bar{\chi}$ denote the endomorphism of Ext (H, G) *induced by an endomorphism χ of G. Then $x \in \operatorname{Im} \bar{\chi}$ if and only if $\alpha(G)/\alpha(\operatorname{Im} \chi)$ is a direct summand of $E/\alpha(\operatorname{Im} \chi)$.*

Proof. Let χ_1 be the homomorphism of G to $\operatorname{Im} \chi$ induced by χ; let π be the restriction to $\alpha(G)$ of the natural projection of E to $E/\alpha(\operatorname{Im} \chi)$; finally, let i_n be the relevant indentity map for $n = 1, 2, 3$.

From the exact sequences

$$0 \longrightarrow \operatorname{Im} \chi \xrightarrow{i_1} G \xrightarrow{\pi\alpha} \alpha(G)/\alpha(\operatorname{Im} \chi) \longrightarrow 0,$$

$$0 \longrightarrow \operatorname{Ker} \chi \xrightarrow{i_2} G \xrightarrow{\chi_1} \operatorname{Im} \chi \longrightarrow 0$$

we get the exact sequences

$$\operatorname{Ext}(H, \operatorname{Im} \chi) \xrightarrow{i_1''} \operatorname{Ext}(H, G) \xrightarrow{(\pi\alpha)''} \operatorname{Ext}[H, \alpha(G)/x(\operatorname{Im} \chi)] \longrightarrow 0,$$

$$\operatorname{Ext}(H, \operatorname{Ker} \chi) \xrightarrow{i_2''} \operatorname{Ext}(H, G) \xrightarrow{\chi_1''} \operatorname{Ext}(H, \operatorname{Im} \chi) \longrightarrow 0.$$

Thus χ_1'' is onto so that $\operatorname{Im} \bar{\chi} = \operatorname{Im}(i_1'' \chi_1'') = \operatorname{Im} i_1'' = \operatorname{Ker}(\pi\alpha)''$. Since $(\pi\alpha)''(x)$ is determined by

$$0 \longrightarrow \alpha(G)\alpha(\operatorname{Im} \chi) \xrightarrow{i_3} E/\alpha(\operatorname{Im} \chi) \xrightarrow{\psi} H \longrightarrow 0$$

where $\psi[e + \alpha(\operatorname{Im} \chi)] = \beta(e)$, Theorem A follows.

Using a similar notation, from the sequences

$$0 \longrightarrow \operatorname{Ker} \theta \xrightarrow{i_1} H \xrightarrow{\theta_1} \operatorname{Im} \theta \longrightarrow 0,$$

$$0 \longrightarrow \operatorname{Im} \theta \xrightarrow{i_2} H \xrightarrow{\pi} H/\operatorname{Im} \theta \longrightarrow 0$$

there follows dually

Theorem B. *Let x denote the element of* Ext(H, G) *determined by the exact sequence*

$$0 \longrightarrow G \xrightarrow{\alpha} E \xrightarrow{\beta} G \longrightarrow 0;$$

let $\bar{\theta}$ denote the endomorphism of Ext (H, G) *induced by an endomorphism $\bar{\theta}$ of H. Then $x \in \operatorname{Im} \bar{\theta}$ if and only if $\alpha(G)$ is a direct summand of* $\beta^{-1}(\operatorname{Ker} \theta)$.

We first obtained Theorem B by using [2, Theorem 1. 2′] and standard properties of divisible groups to directly dualize the proof given in [1] for [1, Theorem 62. 4]. Professor R. S. PIERCE, University of Washington, gave the above elegant homological proof of Theorem B; our proof of Theorem A, in turn, is the dual of his argument.

The author would like to thank Professor R. A. BEAUMONT, University of Washington, for his encouragement in the preparation of this note.

References

[1] L. FUCHS, *Abelian Groups* (Budapest, 1958).
[2] S. MACLANE, Duality for groups, *Bulletin Amer. Math. Soc.*, **56** (1950), 485–516.

(Received September 27, 1962)